

Multivariable Calculus with Lab4IQM

Volume 3, Chapters 2–7: vectors, partial derivatives, multiple integrals,
vector calculus, second-order differential equations

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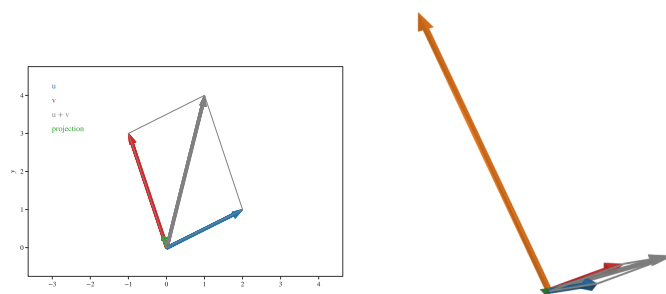
This companion follows the chapters and sections of *Calculus Volume 3* by Gilbert Strang, Edwin Herman and others (OpenStax, Rice University), a free textbook that students can read online or download. Chapter 1 of Volume 3 (parametric equations and polar coordinates) is the same as Chapter 7 of Volume 2: use the Integral Calculus module (`parametric`, `polar`, `polararea`, `conic`). For each section the companion gives the Lab4IQM commands that draw the ideas in 3D and compute the results exactly, with Lab4IQM's symbolic engine; systems of equations are solved numerically and their solutions recognised as exact numbers. Read the textbook for the explanations and the exercises; use Lab4IQM to see the vectors, curves and surfaces and to check your answers. This version covers Chapters 2–7, the whole volume, with practice problems for every chapter.

Type the commands in the Lab4IQM console after `import multicalc` (the module needs Lab4IQM 1.7). Vectors and points are written in parentheses, $\mathbf{u}=(1, 2, 3)$, with exact entries such as `sqrt(2)` or `pi/4`. Turn the 3D pictures with the mouse in Blender's view.

2.1–2.4 Vectors in the Plane and in Space; the Dot and Cross Products

```
vectors(u=(2, 1), v=(-1, 3))          lengths, u + v, u . v, the angle, the  
    projection                          ... and u x v, the area of the  
vectors(u=(3, -1, 2), v=(1, 4, 0))    parallelogram  
    projection                          the triple product: a volume of 2  
vectors(u=(1, 0, 1), v=(0, 1, 1), w=(1, 1, 0))
```

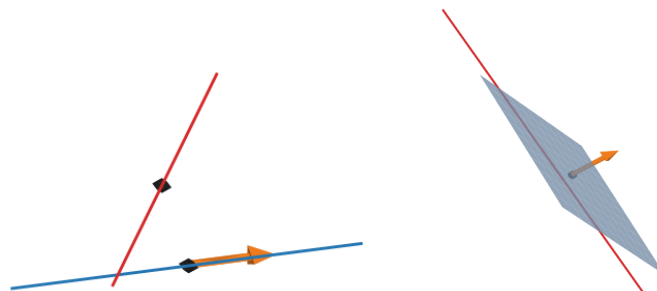
The arrows start at the origin: u blue, v red, $u + v$ grey (the parallelogram), the projection of u on v green, $u \times v$ orange. All results are exact: $\cos \theta = \sqrt{2}/10$, $|u \times v| = \sqrt{237}$.



2.5 Equations of Lines and Planes in Space

```
line3(p=(1, 2, 3), d=(2, -1, 4), point=(0, 0, 0))    equations; distance 5*sqrt  
(14)/7
```

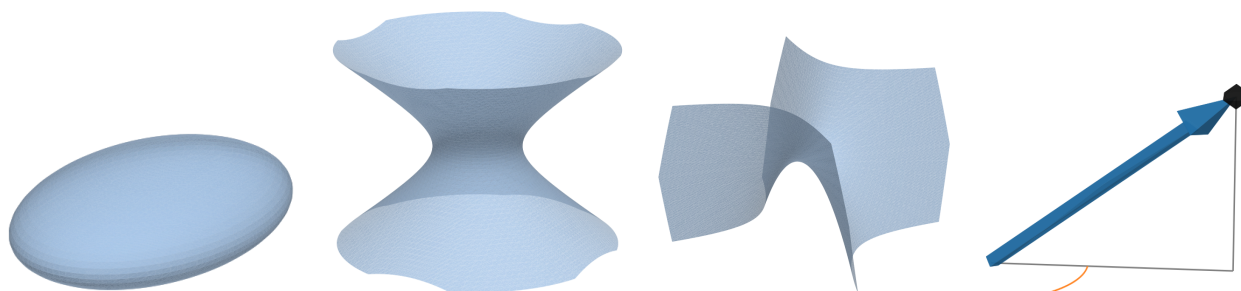
<code>line3(p=(1, 0, 0), d=(1, 1, 0), p2=(0, 1, 1), d2=(1, -1, 2))</code>	skew lines, distance sqrt(3)
<code>plane(p=(1, 0, 0), q=(0, 2, 0), r=(0, 0, 3))</code>	$6x + 3y + 2z = 6$
<code>plane(2*x + 3*y - z = 6, point=(1, 1, 1))</code>	distance sqrt(14)/7
<code>plane(x + y + z = 1, plane2=x - y + 2*z = 4)</code>	the angle, the line of
<code>intersection</code>	
<code>plane(x + y + z = 1, p0=(0, 0, 0), d=(1, 2, 3))</code>	where a line meets the
<code>plane</code>	



2.6–2.7 Quadric Surfaces; Cylindrical and Spherical Coordinates

<code>quadric(x^2/4 + y^2/9 + z^2 = 1)</code>	an ellipsoid
<code>quadric(x^2 + y^2 - z^2 = 1)</code>	a hyperboloid of one sheet
<code>quadric(z^2 - x^2 - y^2 = 1)</code>	... of two sheets
<code>quadric(x^2 + y^2 = z^2)</code>	an elliptic cone
<code>quadric(z = x^2 + y^2)</code>	an elliptic paraboloid
<code>quadric(z = y^2 - x^2)</code>	a hyperbolic paraboloid (a saddle)
<code>quadric(x^2 + z^2 = 4)</code>	a cylinder along the y-axis
<code>coords3(x=1, y=1, z=sqrt(2))</code>	cylindrical (sqrt(2), pi/4, sqrt(2)), spherical (2, pi/4, pi/4)
<code>coords3(rho=4, theta=pi/4, phi=pi/6) and back</code>	

`quadric` completes the squares, reads the signs, names the surface and its axis, and lists the traces in the coordinate planes.



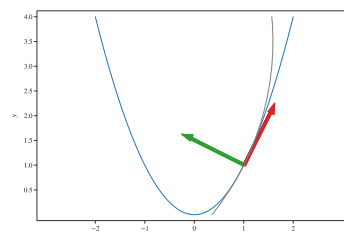
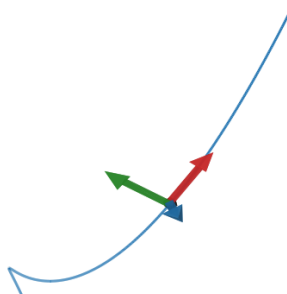
3.1–3.4 Vector-Valued Functions, Arc Length and Curvature, Motion in Space

<code>spacecurve(r=(cos(t), sin(t), t), t=0..4*pi, at=pi/2)</code>	a helix: T, N, B,
<code>curvature 1/2</code>	
<code>spacecurve(r=(t, t^2, t^3), t=-1..1.5, at=1)</code>	the twisted cubic
<code>spacecurve(r=(3*cos(t), 3*sin(t), 4*t), t=0..2*pi, length=true)</code>	$L = 10\pi$, $r(s)$
<code>spacecurve(r=(t, t^2), t=-2..2, at=1)</code>	a plane curve: the
<code>osculating circle</code>	

```
projectile(v0=30, angle=pi/4)
flight
```

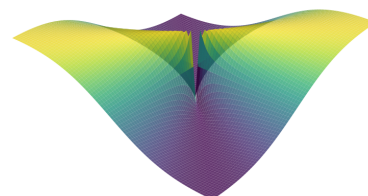
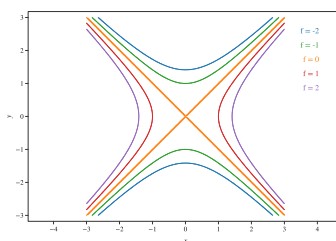
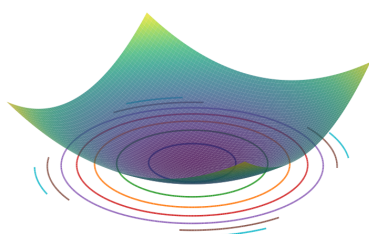
range, height, time of

`spacecurve` gives $r'(t)$, $r''(t)$, the speed, the curvature $\kappa = |r' \times r''|/|r'|^3$ and the tangential and normal components of the acceleration; `at=` draws the unit tangent T (red), normal N (green), binormal B (blue) and the osculating circle of radius $1/\kappa$; `length=true` gives the arc length and, for constant speed, the arc-length parametrization.



4.1–4.2 Functions of Several Variables; Limits and Continuity

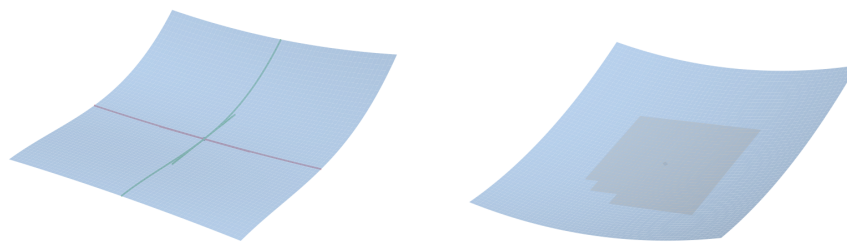
<code>fsurface(f=x^2 + y^2)</code>	the graph and its level curves
<code>fsurface(f=sqrt(9 - x^2 - y^2))</code>	the domain is a disk
<code>levels(f=x^2 - y^2, levels=(-2, -1, 0, 1, 2))</code>	a contour map
<code>limit2(f=x*y/(x^2 + y^2), at=(0, 0))</code>	lines give different values: no limit
<code>limit2(f=x^2*y/(x^4 + y^2), at=(0, 0))</code>	lines agree, the parabola $y = x^2$ does not
<code>limit2(f=x^2*y/(x^2 + y^2), at=(0, 0))</code>	the polar check: the limit is 0



4.3–4.5 Partial Derivatives, Tangent Planes, the Chain Rule

<code>partial(f=x^2*y + 3*y^3, at=(1, 1))</code>	f_x, f_y , second partials,
Clairaut	
<code>partial(f=x*y*z + z^2, at=(1, 2, 3))</code>	three variables
<code>tangentplane(f=x^2 + y^2, at=(1, 2))</code>	$z = 2x + 4y - 5$
<code>tangentplane(f=sqrt(20 - x^2 - 7*y^2), at=(2, 1), dx=0.04, dy=-0.02)</code>	the differential
<code>dz</code>	
<code>tangentplane(x^2 + y^2 + z^2 = 9, at=(1, 2, 2))</code>	a surface $F = c$: the gradient as
normal	
<code>chainrule(f=x^2*y, x=cos(t), y=sin(t))</code>	dz/dt
<code>chainrule(f=exp(x)*sin(y), x=s*t^2, y=s^2*t)</code>	dz/ds and dz/dt
<code>chainrule(x^3 + y^3 = 6*x*y)</code>	implicit: $dy/dx = -F_x/F_y$

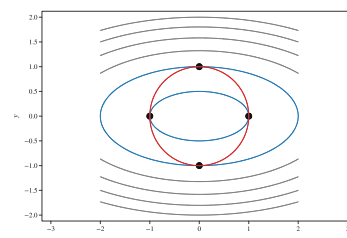
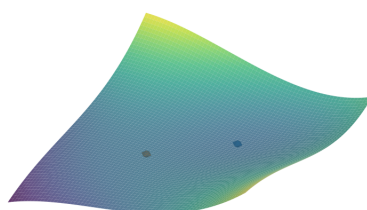
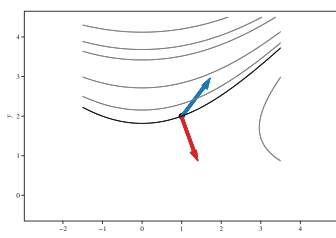
`partial` draws the traces through the point (red: y fixed, slope f_x ; green: x fixed, slope f_y); `tangentplane` draws the plane (orange) touching the surface.



4.6–4.8 Directional Derivatives and the Gradient; Maxima and Minima; Lagrange Multipliers

<code>gradient(f=x^2*y - y^3, at=(1, 2), u=(3, 4))</code>	<code>grad f = <4, -11>, D_u f = -32/5</code>
<code>gradient(f=x*y*z, at=(1, 2, 3), u=(1, 1, 1))</code>	
<code>extrema2(f=x^3 + y^3 - 3*x*y)</code>	a saddle at (0, 0), a minimum at (1, 1)
<code>extrema2(f=x^4 + y^4 - 4*x*y + 1)</code>	
<code>lagrange(f=x^2 + 4*y^2, g=x^2 + y^2 - 1)</code>	on the circle: largest 4, smallest 1
<code>lagrange(f=x + 2*y - z, g=x^2 + y^2 + z^2 - 6)</code>	on a sphere: 6 and -6
<code>lagrange(f=x^2 + y^2 + z^2, g=x + y + z - 1, h=x - y)</code>	two constraints

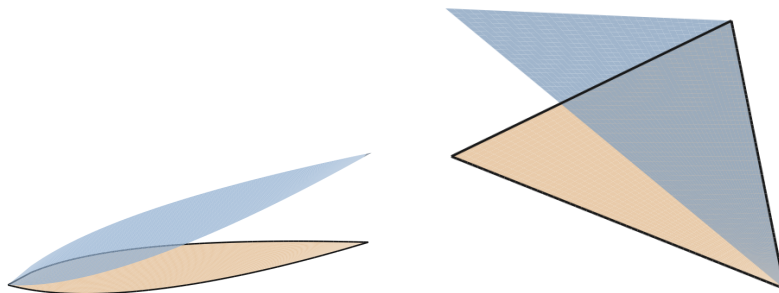
The gradient (red) is perpendicular to the level curve through the point and points the way f increases fastest. `extrema2` solves $f_x = f_y = 0$ and applies the second derivative test with $D = f_{xx}f_{yy} - f_{xy}^2$. `lagrange` solves $\nabla f = \lambda \nabla g$, $g = 0$ and draws the level curves of f tangent to the constraint (red) at the solutions.



5.1–5.2 Double Integrals over Rectangles and General Regions

<code>double(f=x*y^2, x=0..2, y=0..1)</code>	a rectangle: 2/3
<code>double(f=x + y, y=x^2..sqrt(x), x=0..1)</code>	a region of type I: 3/10
<code>double(f=1, y=x^2..2*x, x=0..2)</code>	$f = 1$: the area, 4/3
<code>double(f=exp(y^2), y=x..1, x=0..1, swap=true)</code>	reverse the order: $e/2 - 1/2$
<code>double(f=6 - 2*x - 3*y, x=0..3, y=0..2 - 2*x/3)</code>	a volume under a plane: 6

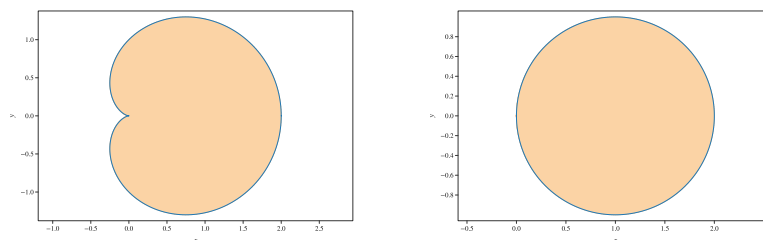
Write the limits as `var=lo..hi`; the inner limits may use the outer variable, in whichever order you type them. Each step of the iterated integral is shown (the inner integral, then the outer one), and the exact value is compared with a numerical value computed independently by Gauss–Legendre quadrature. When no antiderivative is found the value is numerical. `swap=true` describes the same region the other way round, which turns $\int e^{y^2} dy$, impossible in elementary terms, into an easy integral. The region is drawn in the xy -plane (orange) with the surface $z = f(x, y)$ over it (blue).



5.3 Double Integrals in Polar Coordinates

```
polarint(f=x^2 + y^2, r=0..2, theta=0..2*pi)      8 pi
polarint(f=exp(-(x^2 + y^2)), r=0..3, theta=0..pi/2)  the integrand becomes r exp(-r^2)
polarint(f=1, r=0..1 + cos(theta), theta=0..2*pi)    the area of a cardioid: 3 pi/2
polarint(f=sqrt(4 - x^2 - y^2), r=0..2*cos(theta), theta=-pi/2..pi/2)
```

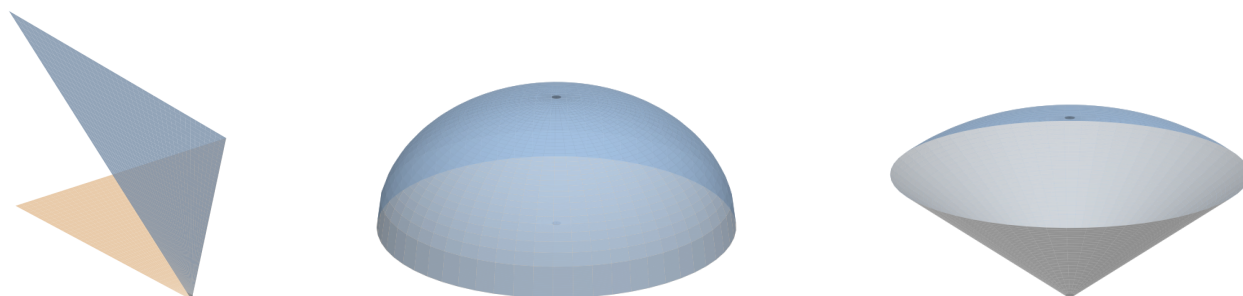
f may be written in x, y : it is rewritten with $x = r \cos \theta$, $y = r \sin \theta$, simplified with $\sin^2 \theta + \cos^2 \theta = 1$ and multiplied by the Jacobian r of $dA = r dr d\theta$.



5.4–5.5 Triple Integrals; Cylindrical and Spherical Coordinates

```
triple(f=x*y*z, x=0..1, y=0..2, z=0..3)          a box: 9/2
triple(f=1, z=0..1 - x - y, y=0..1 - x, x=0..1)  the volume of a tetrahedron: 1/6
triple(f=z, z=0..sqrt(4 - r^2), r=0..2, theta=0..2*pi)  cylindrical: dV = r dz dr dtheta
triple(f=sqrt(x^2 + y^2), z=0..4, r=0..2, theta=0..2*pi)
triple(f=1, rho=0..2, phi=0..pi/4, theta=0..2*pi)      spherical: an ice-cream cone
triple(f=x^2 + y^2 + z^2, rho=0..1, phi=0..pi, theta=0..2*pi)  4 pi/5
```

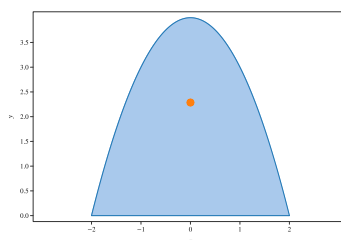
Giving r and θ makes the coordinates cylindrical; ρ , ϕ and θ make them spherical, with $dV = \rho^2 \sin \phi d\rho d\phi d\theta$. The solid's boundary surfaces are drawn: the top or outer surface blue, the bottom or inner one orange, the sides grey.



5.6–5.7 Centres of Mass and Moments of Inertia; Change of Variables

<code>lamina(density=1, y=0..sqrt(x), x=0..4)</code>	the centroid (12/5, 3/4)
<code>lamina(density=x + y, x=0..1, y=0..1)</code>	
<code>lamina(density=y, y=0..4 - x^2, x=-2..2)</code>	the centre of mass (0, 16/7)
<code>jacobian(x=u + v, y=u - v)</code>	$J = -2$
<code>jacobian(x=r*cos(theta), y=r*sin(theta))</code>	$J = r$
<code>jacobian(x=rho*sin(phi)*cos(theta), y=rho*sin(phi)*sin(theta), z=rho*cos(phi))</code>	
<code>jacobian(x=(u + v)/2, y=(u - v)/2, f=x^2 - y^2, u=0..2, v=0..1)</code>	an integral in u, v

`lamina` gives the mass, the moments M_x , M_y , the centre of mass (orange), the moments of inertia I_x , I_y , I_0 and the radii of gyration. `jacobian` gives the matrix of partial derivatives and $J = \partial(x, y)/\partial(u, v)$; with `f=` and the limits in the new variables it computes $\iint f dA = \iint f(x(u, v), y(u, v)) |J| du dv$.



6.1, 6.3, 6.5 Vector Fields, Conservative Fields, Divergence and Curl

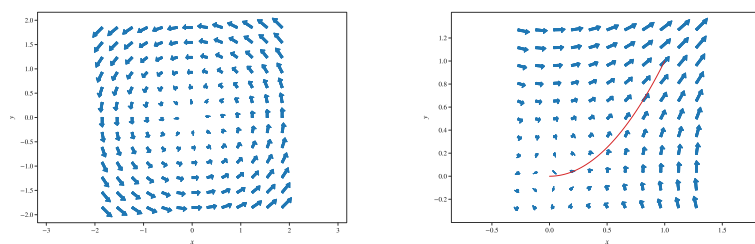
<code>vfield(F=(-y, x))</code>	a rotation: curl 2, not conservative
<code>vfield(F=(2*x*y, x^2 + 3*y^2))</code>	conservative: $f = x^2 y + y^3$
<code>vfield(F=(y*z, x*z, x*y))</code>	in space: $f = xyz$
<code>vfield(F=(y, -x, z), x=-1..1, y=-1..1, z=-1..1)</code>	$\text{div } F = 1, \text{curl } F = \langle 0, 0, -2 \rangle$

`vfield` draws the field as arrows and gives $\text{div } F$ and $\text{curl } F$ (for a plane field the scalar curl $Q_x - P_y$). When the cross-partial agree, a potential function is built one variable at a time: $f = \int P dx + g(y, z)$, then g from Q , then from R , and $\nabla f = F$ is checked.

6.2–6.3 Line Integrals; the Fundamental Theorem for Line Integrals

<code>lineint(f=x*y, r=(cos(t), sin(t)), t=0..pi/2)</code>	with respect to arc length: 1/2
<code>lineint(f=1, r=(3*cos(t), 3*sin(t), 4*t), t=0..2*pi)</code>	$f = 1$: the length, 10 pi
<code>lineint(F=(-y, x), r=(cos(t), sin(t)), t=0..2*pi)</code>	circulation: 2 pi
<code>lineint(F=(y, x), r=(t, t^2), t=0..1)</code>	work; also $f(\text{end}) - f(\text{start})$
<code>lineint(F=(x, y, z), r=(cos(t), sin(t), t), t=0..2*pi)</code>	along a helix: 2 pi^2
<code>lineint(F=(x, y), r=(cos(t), sin(t)), t=0..2*pi, flux=true)</code>	flux across the circle

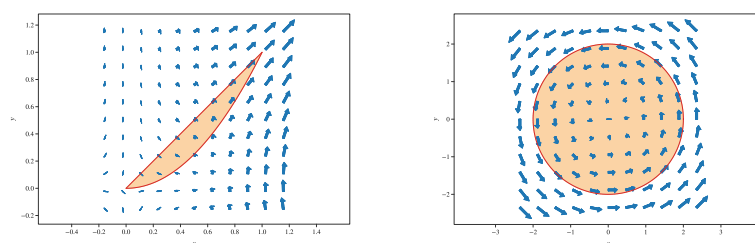
With `f=` the integral is $\int f(r(t)) |r'(t)| dt$; with `F=` it is $\int F(r(t)) \cdot r'(t) dt$, and when F is conservative the value is confirmed by $f(r(b)) - f(r(a))$, the same for every path.



6.4 Green's Theorem

<code>green(F=(-y, x), x=0..1, y=0..1)</code>	circulation around a square: 2
<code>green(F=(x*y, x^2), y=x^2..x, x=0..1)</code>	1/12
<code>green(F=(-y, x), r=0..2, theta=0..2*pi)</code>	a disk: 8 pi
<code>green(F=(y^2, 3*x*y), y=0..sqrt(1 - x^2), x=-1..1)</code>	a half disk: 2/3
<code>green(F=(x, y), r=0..1, theta=0..2*pi, flux=true)</code>	the flux form: 2 pi

Both sides of the theorem are computed separately: the double integral of $Q_x - P_y$ (or of $\text{div } F$ in the flux form) exactly, step by step, and the line integral around the boundary, counterclockwise, numerically along each piece of the boundary. They agree.



6.6 Surface Integrals

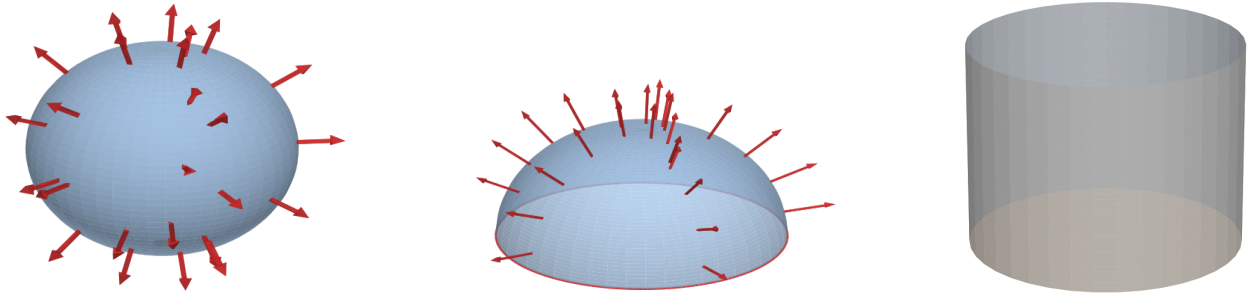
<code>surfint(f=1, r=(2*sin(v)*cos(u), 2*sin(v)*sin(u), 2*cos(v)), u=0..2*pi, v=0..pi)</code>	
<code>surfint(f=1, r=(u*cos(v), u*sin(v), u), u=0..1, v=0..2*pi)</code>	a cone: $\sqrt{2}$ pi
<code>surfint(f=z, z=4 - x^2 - y^2, x=0..1, y=0..1)</code>	a graph $z = g(x, y)$
<code>surfint(F=(x, y, z), r=(sin(v)*cos(u), sin(v)*sin(u), cos(v)), v=0..pi, u=0..2*pi)</code>	
<code>surfint(F=(0, 0, z), z=1 - x - y, y=0..1 - x, x=0..1)</code>	flux upward: 1/6

The sphere of radius 2 has area 16π ; the flux of (x, y, z) out of the unit sphere is 4π , and Stokes' theorem on the upper half of the unit sphere gives π . The normal is $r_u \times r_v$, with the parameters in the order you type them; on a closed surface the output says whether it points outward or inward (`flip=true` reverses it). On a graph $z = g(x, y)$ the normal points up. The unit normals are drawn in red.

6.7–6.8 Stokes' Theorem; the Divergence Theorem

<code>stokes(F=(-y, x, z), z=4 - x^2 - y^2, r=0..2, theta=0..2*pi)</code>	8 pi
<code>stokes(F=(z, x, y), r=(sin(v)*cos(u), sin(v)*sin(u), cos(v)), v=0..pi/2, u=0..2*pi)</code>	
<code>stokes(F=(y, -x, x*z), z=1 - x - y, y=0..1 - x, x=0..1)</code>	-7/6
<code>divthm(F=(x, y, z), x=0..1, y=0..1, z=0..1)</code>	3
<code>divthm(F=(x^3, y^3, z^3), rho=0..1, phi=0..pi, theta=0..2*pi)</code>	12 pi/5
<code>divthm(F=(x, y, z^2), z=0..2, r=0..1, theta=0..2*pi)</code>	8 pi

`stokes` computes the flux of $\text{curl } F$ through the surface exactly and the circulation of F around the boundary curve (red), oriented with the normal by the right-hand rule, numerically. `divthm` computes the triple integral of $\text{div } F$ exactly and the outward flux through every face of the boundary numerically. Seams of a parametrization (such as $\theta = 0$ and $\theta = 2\pi$) are traced twice in opposite directions and cancel, and faces that collapse to a curve or a point contribute nothing.



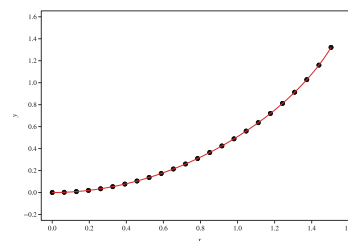
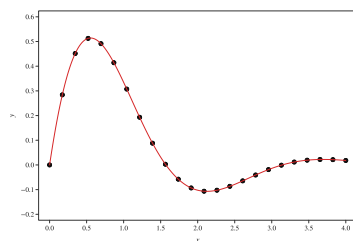
7.1–7.2 Second-Order Linear Equations; Nonhomogeneous Equations

```
ode2(a=1, b=-3, c=2)
ode2(a=1, b=4, c=4, y0=1, dy0=0)
  problem
ode2(a=1, b=2, c=5, y0=0, dy0=2)
ode2(a=1, b=0, c=4, y0=0, bc=(pi/4, 3))
ode2(a=1, b=-1, c=-2, f=4*x^2, y0=0, dy0=1)
ode2(a=1, b=0, c=4, f=3*sin(2*x))
ode2(a=1, b=0, c=1, f=sec(x), y0=0, dy0=0)
```

two real roots: $C_1 e^{(2x)} + C_2 e^{-x}$
a repeated root, an initial value

complex roots: $e^{-x} \sin(2x)$
a boundary value problem
undetermined coefficients
the guess multiplied by x
variation of parameters

`ode2` solves $ay'' + by' + cy = f(x)$. It shows the characteristic equation and its roots, the homogeneous solutions, the guess for the particular solution (multiplied by x or x^2 when needed) and the coefficients found by substituting it, or, when f is not of that kind, the Wronskian and u'_1 , u'_2 of variation of parameters. The conditions fix C_1 and C_2 . The solution (red) is compared with a Runge–Kutta solution computed independently (black dots).

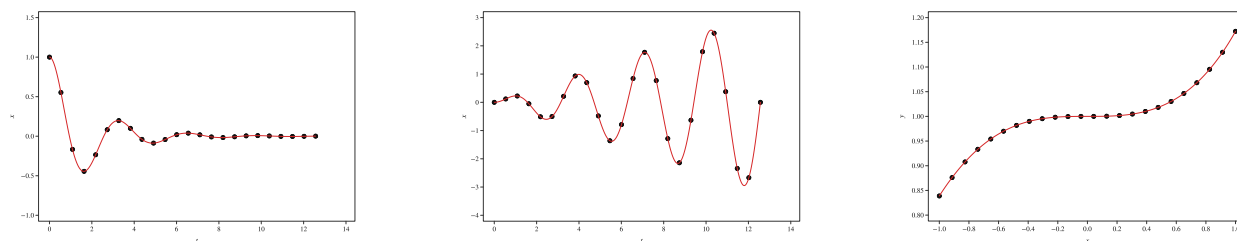


7.3–7.4 Applications; Series Solutions of Differential Equations

```
spring(m=1, k=4, x0=1)
spring(m=1, k=4, c=1, x0=1)
spring(m=1, k=4, c=4, x0=1)
spring(m=1, k=4, c=5, x0=1)
spring(m=1, k=4, F=cos(2*t))
rlc(L=1, R=2, C=1/5, E=10)
odeseries(P=1, Q=0, R=1)
odeseries(P=1, Q=0, R=-x, y0=1, dy0=0)
odeseries(P=1 - x^2, Q=-2*x, R=6, n=10)
  solution
```

simple harmonic motion, period π
underdamped
critically damped
overdamped
resonance: $x = t \sin(2t)/4$
the charge and the current
the series of $\cos x$ and $\sin x$
Airy's equation $y'' - xy = 0$
Legendre's equation: a polynomial

spring classifies the motion by $c^2 - 4mk$ and detects resonance; **rlc** is the same equation with $m = L$, $c = R$, $k = 1/C$. **odeseries** matches the coefficient of each power of x in $P y'' + Q y' + R y = 0$ to find a_{k+2} from the earlier coefficients, as exact fractions, and draws the partial sum against a Runge–Kutta solution.



Practice problems and checking your answers

```
practice(double)                    five problems on a topic
practice(potential, n=8, seed=3)    the same seed gives the same problems
practice(nonhomogeneous, answers=true) with the answers
check(double, f=x*y, x=0..1, y=0..x, answer=1/8)
check(cross, u=(1, 0, 0), v=(0, 1, 0), answer=(0, 0, 1))
check(potential, F=(2*x*y, x^2), answer=x^2*y)
check(ode2, a=1, b=0, c=4, answer=C1*cos(2*x) + C2*sin(2*x))
```

The problems are made up at random by the module (they are not the textbook's exercises), with answers computed by the module itself. Topics: **dot**, **cross**, **planes**, **quadrics** (Ch. 2); **arclength3** (Ch. 3); **partials**, **tangentplanes**, **directional**, **extrema** (Ch. 4); **double**, **polar**, **triple**, **jacobians** (Ch. 5); **potential**, **lineintegrals**, **divcurl**, **green**, **flux** (Ch. 6); **homogeneous**, **nonhomogeneous**, **ivp**, **springs** (Ch. 7). **check** accepts any equivalent form of the answer (a potential function up to a constant, a general solution with any two independent solutions) and gives a hint when the answer is wrong: the order in a cross product, a forgotten Jacobian r or $\rho^2 \sin \phi$, the wrong partial derivative, a direction that is not a unit vector, two solutions that are not independent.

How the results are checked

The tests check the vector operations (the cross product is orthogonal to both vectors, exact angles and areas), lines (skew, parallel, intersecting, distances) and planes (from points, equations, distances, intersections), the classification of every kind of quadric surface, coordinate conversions, the speed, curvature and the TNB frame of the helix and other curves, arc lengths and arc-length parametrizations, projectiles, limits in two variables (by paths and in polar form), partial derivatives and Clairaut's theorem, tangent planes and differentials, the chain rule against direct differentiation, gradients and directional derivatives, critical points with the second derivative test, Lagrange multipliers with one and two constraints, double and triple integrals in every coordinate system (each exact value against an independent numerical one), reversing the order of integration, centres of mass, Jacobians, potential functions, line integrals against the Fundamental Theorem, surface areas and fluxes, both sides of the theorems of Green, Stokes and Gauss (computed independently), second-order equations against substitution and a Runge–Kutta solution, series coefficients, every practice topic (each generated answer is accepted by **check**), and the console output of every command.

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