

Differential Calculus with Lab4IQM

Chapters 1–4: functions, limits, derivatives and their applications

Lab4IQM Differential Calculus module 0.2 · lab4iqm.com

This companion follows the chapters and sections of *Calculus Volume 1* by Gilbert Strang, Edwin Herman and others (OpenStax, Rice University), a free textbook that students can read online or download. For each section it gives the Lab4IQM commands that draw the ideas of the section and compute its results exactly: formulas, limits and derivatives come from Lab4IQM's own symbolic engine and are checked numerically. Read the textbook for the explanations and the exercises; use Lab4IQM to see the graphs, test your intuition, and check your answers. This version covers Chapters 1–4 (the integration chapters 5–6 of Volume 1 open the Integral Calculus module, which follows Volume 2; the Multivariable Calculus module follows Volume 3).

Type the commands in the Lab4IQM console after `import diffcalc`. `help(derivative)` explains any command; the Examples panel has one example per section; `view2d()` looks at a graph straight on. Every command that draws also prints its results in the console.

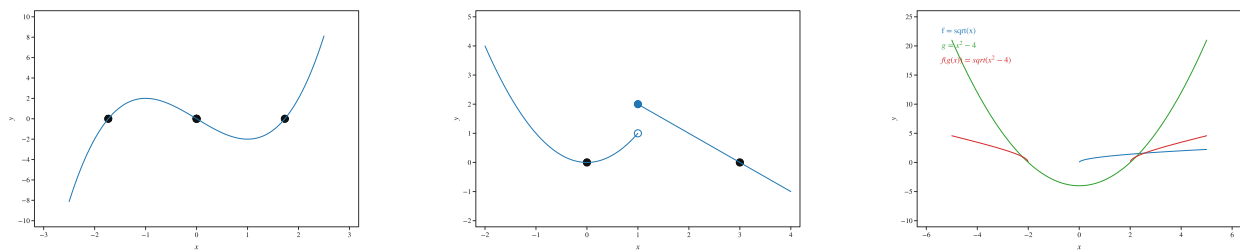
Typing functions

<code>f=x^3 - 3*x, f=2x^2</code>	powers with <code>^</code> ; <code>2x</code> means <code>2*x</code>
<code>sqrt(x), cbrt(x), x^(2/3)</code>	roots (odd roots of negative numbers are real)
<code>sin cos tan sec csc cot, asin acos atan</code>	trigonometric functions (radians) and inverses
<code>exp(x), e^x, log(x) or ln(x), log10, log2</code>	exponential and logarithms
<code>sinh cosh tanh, abs floor ceil sign</code>	hyperbolic and piecewise functions
<code>pi, e</code>	the constants (kept exact: <code>a=pi/4</code>)
<code>f="x^2 for x < 1; 3 - x for x >= 1"</code>	a piecewise function (quoted)
<code>f=a*x^2 + b, a=2, b=-1</code>	other letters are parameters, given as options
<code>x=-2..4, ylim=-10..10</code>	the window

1.1 Review of Functions

<code>function(f=x^3 - 3*x)</code>	class, domain, zeros, symmetry, increasing/decreasing
<code>function(f=sqrt(4 - x^2))</code>	domain <code>[-2, 2]</code> , range <code>[0, 2]</code>
<code>function(f="x^2 for x < 1; 3 - x for x >= 1", x=-2..4, at=1)</code>	
<code>compose(f=sqrt(x), g=x^2 - 4, at=3)</code>	<code>f(g(x))</code> , <code>g(f(x))</code> and their domains

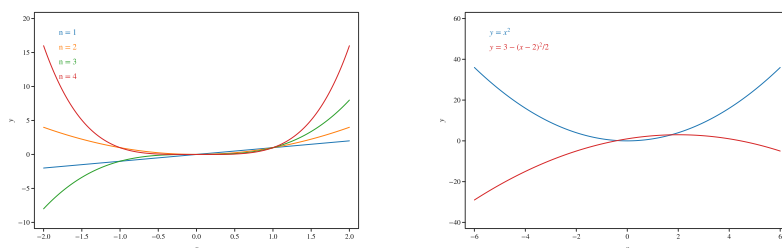
`function` draws $y = f(x)$ and describes it: the conditions on the domain read from the formula (denominators $\neq 0$, even roots ≥ 0 , logarithms > 0), solved to interval notation when they are polynomial; the zeros and the y -intercept (black dots); the range on the window; whether f is even or odd; the intervals where it increases or decreases. Piecewise functions show open and closed ends. `compose` builds both composites and shows that composition is not commutative.



1.2 Basic Classes of Functions

<code>function(f=2*x^4 - 3*x + 1)</code> <code>end behaviour</code>	polynomial: degree, leading coefficient,
<code>function(f=(x^2 - 1)/(x - 2), ylim=-20..20)</code>	rational
<code>family(f=x^n, n=(1, 2, 3, 4), x=-2..2)</code>	a family of power functions
<code>transform(f=x^2, a=-1/2, h=2, k=3)</code>	$y = a f(b(x - h)) + k$, step by step

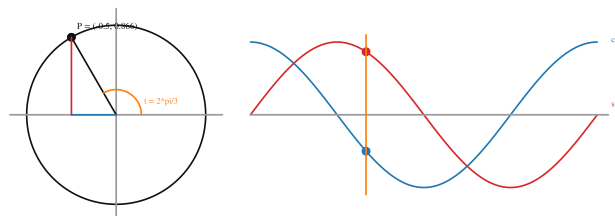
The console names the class of each function: polynomial (with its degree and end behaviour), rational, algebraic, trigonometric, exponential, logarithmic, transcendental or piecewise. **family** overlays the members of a family of functions; **transform** lists the shifts, stretches and reflections in order.



1.3 Trigonometric Functions

<code>unitcircle(t=pi/6)</code>	<code>unitcircle(t=2*pi/3)</code>	<code>unitcircle(t=5*pi/4)</code>
<code>function(f=tan(x))</code>	<code>family(f=sin(k*x), k=(1, 2, 3), x=0..2*pi)</code>	

unitcircle shows the angle t on the unit circle, the point $(\cos t, \sin t)$ and its triangle, next to the graphs of sine and cosine. The six trigonometric values are exact for the multiples of $\pi/6$ and $\pi/4$; the console also gives the angle in degrees, the quadrant and the reference angle. **function** reports the period.

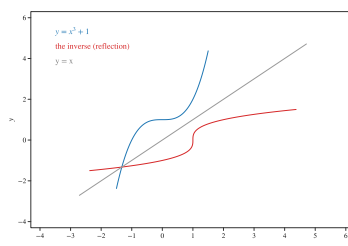


1.4 Inverse Functions

<code>inverse(f=2*x + 3)</code>	<code>inverse(f=x^3 + 1)</code>	<code>inverse(f=(2*x + 1)/(x - 3), x=4..10)</code>
<code>inverse(f=x^2)</code>	the horizontal line test fails; restrict the domain:	

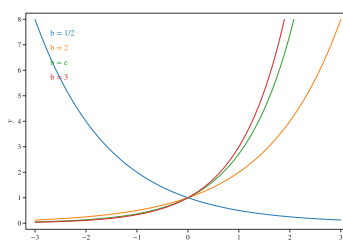
```
inverse(f=x^2, x=0..3)      inverse(f=sin(x), x=-pi/2..pi/2)
```

`inverse` applies the horizontal line test on the interval, finds the inverse by undoing the steps of the formula in reverse order, checks $f^{-1}(f(x)) = x$, and draws f and its reflection in the line $y = x$.



1.5 Exponential and Logarithmic Functions

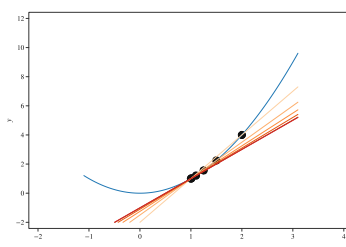
```
family(f=b^x, b=(1/2, 2, e, 3), x=-3..3, ylim=0..8)
family(f=log(x)/log(b), b=(2, e, 10), x=0.05..10)
inverse(f=exp(x))          function(f=log(x^2 - 1))
```



2.1 A Preview of Calculus

```
secant(f=x^2, a=1)          secant(f=sqrt(x), a=4, h=(2, 1, 0.5, 0.1))
```

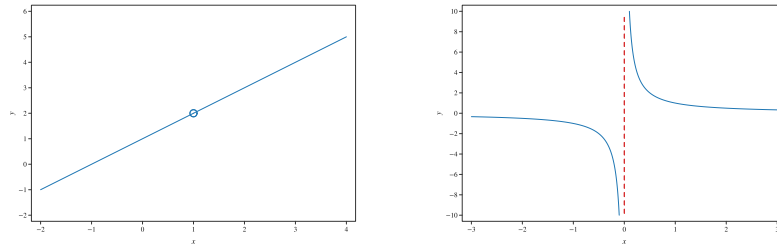
Secant lines through $(a, f(a))$ and $(a + h, f(a + h))$ for smaller and smaller h ; their slopes approach the slope of the tangent line (red), which is given exactly.



2.2 The Limit of a Function

```
limit(f=sin(x)/x, a=0)      a table of values from both sides
limit(f=1/x, a=0)           one-sided infinite limits; the limit does not exist
limit(f=1/(x - 2)^2, a=2)    an infinite limit: a vertical asymptote
limit(f=sin(1/x), a=0)      oscillation: no limit
limit(f="x^2 for x < 1; 3 - x for x >= 1", a=1)
```

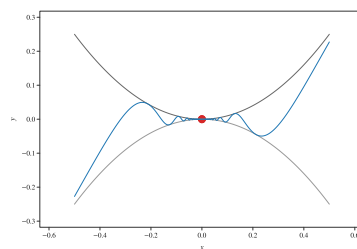
`limit` prints values of f at $a \mp 0.1, 0.01, \dots$ (the numerical evidence of the section), the one-sided limits and the limit, and draws the graph near a : an open circle at the limit point when $f(a)$ is undefined or different, dashed vertical asymptotes. `side=left` or `side=right` asks for one side.



2.3 The Limit Laws

```
limit(f=(x^2 - 1)/(x - 1), a=1)      cancel the common factor (x - 1)
limit(f=(sqrt(x + 4) - 2)/x, a=0)    1/4
limit(f=x^2, a=pi)                   direct substitution: pi^2
squeeze(f=x^2*sin(1/x), lower=-x^2, upper=x^2, a=0)
squeeze(f=sin(x)/x, lower=cos(x), upper=1, a=0, x=-1.5..1.5)
```

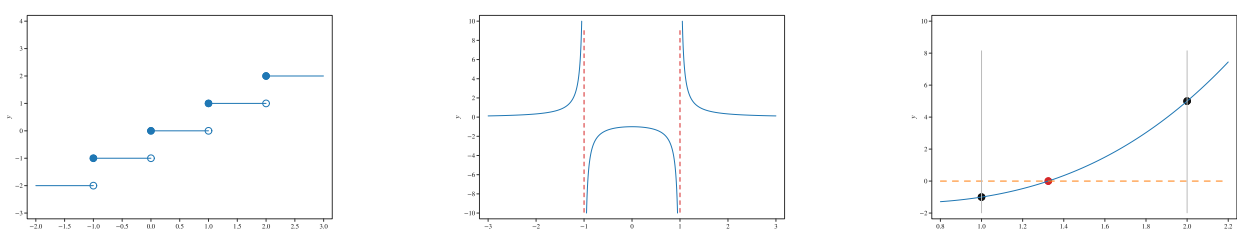
The console says how each limit was found: by direct substitution (a continuous function), by cancelling a common factor of a rational function, by L'Hôpital's rule (Section 4.8), or numerically, with the exact value recognised ($\pi/2$, e , $\sqrt{2}$, fractions). `squeeze` checks the hypotheses of the squeeze theorem and draws the bounds.



2.4 Continuity

```
continuity(f=(x^2 - 4)/(x - 2), x=-1..5)    removable
continuity(f=floor(x), x=-2..3)             four jump discontinuities
continuity(f=1/(x^2 - 1), x=-3..3)          two infinite discontinuities
continuity(f="x + 1 for x < 2; x^2 for x >= 2", x=-1..4)
ivt(f=x^3 - x - 1, a=1, b=2)                the intermediate value theorem and bisection
```

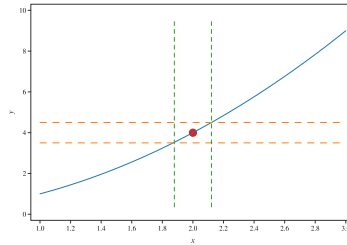
`continuity` applies the three conditions of continuity at every candidate point and classifies the discontinuities. `ivt` checks the hypotheses of the intermediate value theorem and finds c by bisection, listing the intervals.



2.5 The Precise Definition of a Limit

```
epsdelta(f=2*x + 1, a=3, eps=0.5)      delta = eps/2 for a line of slope 2
epsdelta(f=x^2, a=2, eps=0.5)          epsdelta(f=sqrt(x), a=4, eps=0.1)
```

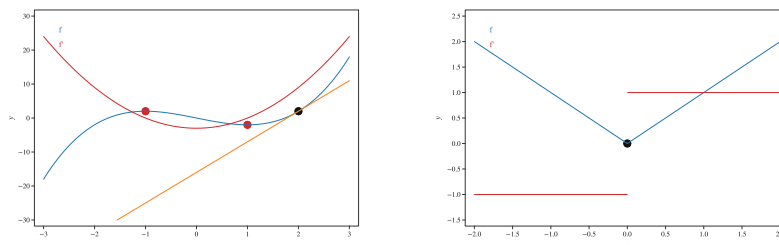
For a given ε , the largest δ with $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$, drawn as two bands: inside the green band the graph stays inside the orange one.



3.1–3.2 Defining the Derivative; The Derivative as a Function

```
derivative(f=x^2, method=definition)      (f(x+h) - f(x))/h = h + 2x, then h -> 0
derivative(f=1/x, method=definition)
derivative(f=x^3 - 3*x, a=2)              f'(a), the tangent and normal lines
derivative(f=abs(x), a=0)                  a corner: not differentiable
derivative(f=cbrt(x), a=0)                 a vertical tangent
derivative(f="x^2 for x < 1; 2*x - 1 for x >= 1")
```

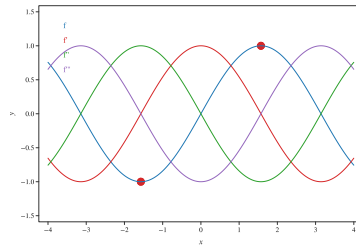
With `method=definition` the difference quotient is simplified until h cancels and then $h \rightarrow 0$. With `a=` the one-sided limits of the difference quotient decide differentiability. The graph shows f (blue), f' (red), the tangent line (orange) and the points where $f' = 0$.



3.3, 3.5, 3.6, 3.9 Differentiation Rules; Trigonometric Functions; The Chain Rule; Exponential and Logarithmic Functions

```
derivative(f=x^3*sin(x))                  rules used: product rule, power rule, d sin u = cos u
  du
derivative(f=x/(x + 1))                   the quotient rule
derivative(f=exp(-x^2), a=1)               the chain rule
derivative(f=sin(x), n=3)                  higher derivatives
derivative(f=x^x, method=log)              logarithmic differentiation: ln y = x ln x
```

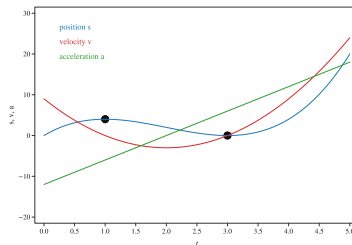
`derivative` shows the rules it used, outermost first, then the simplified derivative.



3.4 Derivatives as Rates of Change

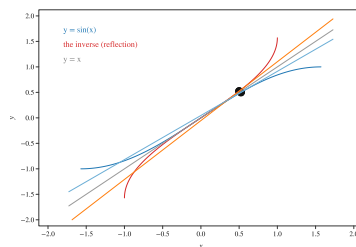
```
motion(s=t^3 - 6*t^2 + 9*t, t=0..5)
motion(s=-4.9*t^2 + 20*t + 2, t=0..4.5, at=1)    a ball thrown upwards
```

Position, velocity and acceleration; the times at rest, the direction of motion, when the particle speeds up or slows down, the displacement and the total distance travelled.



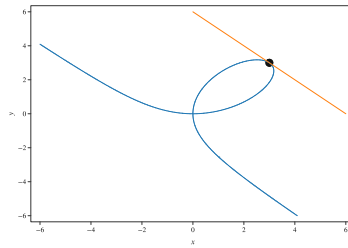
3.7 Derivatives of Inverse Functions

```
inverse(f=sin(x), x=-pi/2..pi/2, at=1/2)    (f^(-1))'(1/2) = 1/f'(pi/6) = 2/sqrt(3)
derivative(f=asin(x), x=-1..1)               derivative(f=atan(x^2))
```



3.8 Implicit Differentiation

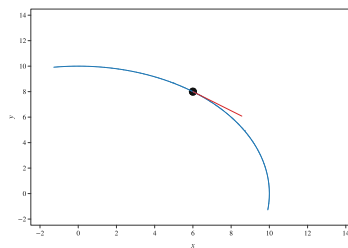
```
implicitdiff(x^2 + y^2 = 25, at=(3, 4))    dy/dx = -x/y; tangent y = 25/4 - 3x/4
implicitdiff(x^3 + y^3 = 6*x*y, at=(3, 3)) the folium of Descartes
implicitdiff(x^2 + y^2 = 25, at=(5, 0))    a vertical tangent
```



4.1 Related Rates

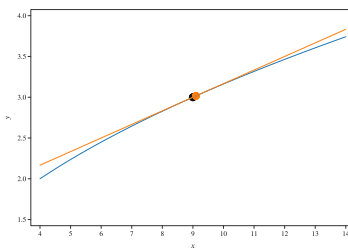
<code>relatedrates(x^2 + y^2 = 100, x=6, y=8, dx=1, find=dy)</code>	a sliding ladder: $dy/dt = -3/4$
<code>relatedrates(V = 4/3*pi*r^3, r=5, dV=100, find=dr)</code>	an inflating balloon: $dr/dt = 1/\pi$
<code>relatedrates(V = pi/3*(2/5)^2*h^3, h=3, dV=2, find=dh)</code>	water filling a cone

Every variable is a function of time; differentiating the equation with respect to t gives one term $\partial F / \partial v \cdot dv/dt$ per variable. The console shows that equation, substitutes the values at the instant, and solves for the unknown rate. A variable left out is found from the equation. For two variables the curve of possible states is drawn with the direction of motion.



4.2 Linear Approximations and Differentials

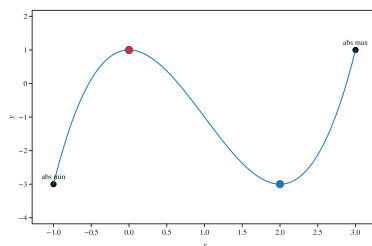
<code>linearize(f=sqrt(x), a=9, at=9.1)</code>	$L(x) = x/6 + 3/2$, error $4.6e-05$
<code>linearize(f=sin(x), a=0, at=0.2, n=2)</code>	with the quadratic approximation
<code>linearize(f=x^3, a=2, dx=0.1)</code>	$dy = f'(a) dx$ versus the actual change;
relative error	



4.3 Maxima and Minima

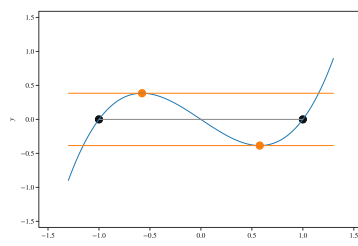
<code>extrema(f=x^3 - 3*x^2 + 1, x=-1..3)</code>	local and absolute extrema; ties are all listed
<code>extrema(f=x^(2/3), x=-1..8)</code>	a critical number where f' is undefined
<code>extrema(f=x*exp(-x), x=0..4)</code>	<code>extrema(f=sin(x) + cos(x), x=0..2*pi)</code>

Critical numbers (where $f' = 0$ or f' is undefined), the first and second derivative tests, and the closed interval method: the absolute extrema are the largest and smallest values of f at the critical numbers and the end points.



4.4 The Mean Value Theorem

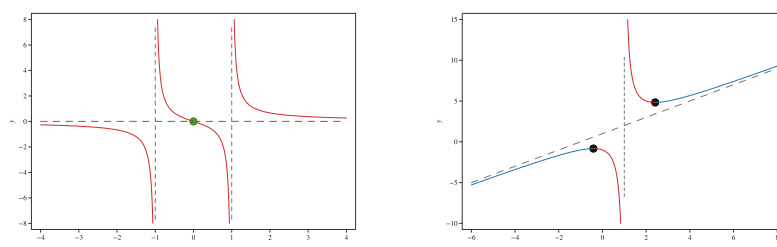
<code>mvt(f=x^2, a=0, b=2)</code>	<code>mvt(f=sqrt(x), a=0, b=9)</code>	$c = 9/4$
<code>mvt(f=x^3 - x, a=-1, b=1)</code>	Rolle's theorem: $c = -\sqrt{3}/3, \sqrt{3}/3$	
<code>mvt(f=abs(x), a=-1, b=2)</code>	the hypotheses fail and there is no c	



4.5 Derivatives and the Shape of a Graph

<code>sketch(f=x^3 - 3*x)</code>	<code>sketch(f=x^4 - 4*x^3)</code>	<code>sketch(f=x^2*exp(-x), x=-1..8)</code>
<code>sketch(f=x/(x^2 - 1), x=-4..4, derivatives=true)</code>		

Everything a sketch needs: domain, intercepts, where f increases (blue) or decreases (red), local extrema, concavity from the sign of f'' , inflection points (green), asymptotes (dashed).



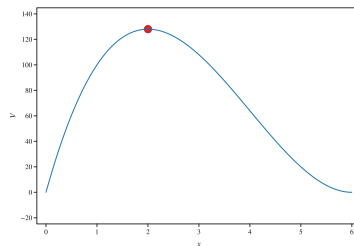
4.6 Limits at Infinity and Asymptotes

<code>asymptotes(f=(2*x^2 + 1)/(x^2 - 4))</code>	vertical $x = -2, 2$ and horizontal $y = 2$
<code>asymptotes(f=(x^2 + 1)/(x - 1))</code>	an oblique asymptote $y = x + 1$
<code>asymptotes(f=atan(x))</code>	different horizontal asymptotes at $+\infty$ and $-\infty$
<code>limit(f=(3*x^2 + 1)/(2*x^2 - x), a=inf)</code>	

4.7 Applied Optimization Problems

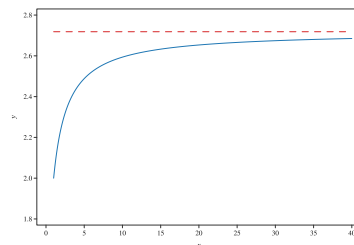
```
optimize(f=x*(12 - 2*x)^2, x=0..6)          a closed interval
optimize(f=2*pi*x^2 + 2000/x, x=0..inf, goal=min) an infinite interval
optimize(problem=box, a=20)          optimize(problem=can)          optimize(problem=closest, a
=2)
```

`problem=` sets up classic problems (box, fence, river, can, window, closest) with their objective functions and intervals; `a=` changes the data. The candidates (critical numbers, end points, limits at open or infinite ends) are listed before the answer.



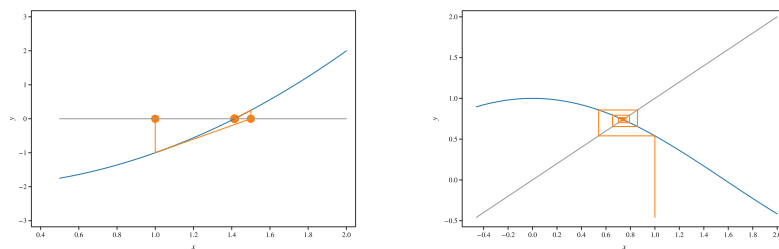
4.8 L'Hôpital's Rule

```
lhospital(f=sin(x)/x, a=0)          0/0
lhospital(f=(exp(x) - 1 - x)/x^2, a=0)  0/0 twice
lhospital(f=x*log(x), a=0, side=right)  0*inf, rewritten as log(x)/(1/x)
lhospital(f=1/x - 1/sin(x), a=0)      inf - inf, combined into one fraction
lhospital(f=(1 + 1/x)^x, a=inf)        1^inf through ln y = x ln(1 + 1/x): the
    limit is e
lhospital(f=(x^2 + 1)/(x + 1), a=1)    not indeterminate: the rule does not apply
```



4.9 Newton's Method

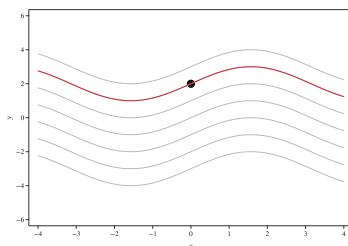
```
newton(f=x^2 - 2, x0=1)          sqrt(2) in five steps
newton(f=cos(x) - x, x0=1)
newton(f=x^3 - 2*x + 2, x0=0)    the iterates cycle 0, 1, 0, 1, ...
newton(f=cbrt(x), x0=1)          the iterates run away
newton(g=cos(x), x0=1)          fixed-point iteration x_(n+1) = g(x_n), drawn as a
    cobweb
```



4.10 Antiderivatives

<code>antiderivative(f=3*x^2 - 4*x + 5)</code>	$x^3 - 2x^2 + 5x + C$
<code>antiderivative(f=cos(x), x0=0, y0=2)</code>	the initial-value problem: $y = \sin(x) + 2$
<code>antiderivative(f=1/x)</code>	$\log(\text{abs}(x)) + C$
<code>antiderivative(a=-9.8, t0=0, v0=20, s0=2)</code>	velocity and position from the acceleration

Antiderivatives come from the derivative rules read backwards (and a linear change of variable and the chain rule in reverse), checked by differentiation. The graph shows the family $F + C$ and the solution of the initial-value problem (red).



Practice and checking your answers

<code>practice(chain)</code>	five problems on the chain rule
<code>practice(product, n=8, seed=3)</code>	the same seed gives the same problems
<code>practice(limits, answers=true)</code>	with the answers
<code>check(derivative, f=sin(3*x), answer=cos(3*x))</code>	
Not quite ... Hint: the correct answer is 3 times yours ... (chain rule)	
<code>check(limit, f=sin(3*x)/x, a=0, answer=3)</code>	
<code>check(inverse, f=2*x + 3, answer=(x - 3)/2)</code>	
<code>check(tangent, f=x^2, a=1, answer=2*x - 1)</code>	
<code>check(antiderivative, f=cos(x), answer=sin(x) + C)</code>	

Topics: domain, composition, inverse, limits, infinity, tangent, power, product, quotient, trig, chain, explog, inversetrig, implicit, mixed, linearization, extrema, lhopital, antiderivative. `check` accepts any equivalent form of an answer (it compares at many points) and names common mistakes: a wrong sign, a missing constant factor from the chain rule, an answer off by a constant. A teacher can give each student a different seed.

How the results are checked

The module's tests check the symbolic engine against finite differences for 35 functions (every rule and every function), the printing round trip, exact values ($\sin(\pi/6) = 1/2$, $8^{2/3} = 4$, $\sqrt{12} = 2\sqrt{3}$), polynomial division and cancellation, Taylor polynomials, and 26 limits (two-sided, one-sided, infinite, at infinity, non-existent). They check inverse functions by composition, domains in interval

notation, piecewise functions, the difference quotient, that every generated practice problem has an answer and that the derivative answers agree with finite differences, and the console output of every command. For Chapter 4 they check related rates, linear approximations, absolute extrema with ties, the mean value theorem, inflection points and asymptotes, the optimization problems, nine limits by L'Hôpital's rule step by step, Newton's method (convergence, a cycle, fixed-point iteration), and antiderivatives (37 forms in the engine, each checked by differentiation).